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Estimation of the Rate of a Doubly Stochastic Time–Space Poisson Process

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We consider the problem of estimating the rate of a doubly stochastic, time–space Poisson process when the observations are restricted to a region $D \subseteq \mathbb{R}^2$, and assuming that the rate process has a Gaussian form. In the case $D = \mathbb{R}^2$, we extend a known result to compute the minimum-mean-square-error (MMSE) estimate explicitly. When $D \neq \mathbb{R}^2$, we consider the use of linear estimates. We give closed-form expressions for the mean and the covariance of the rate process in terms of the mean and the covariance of an underlying state process. This enables us to write down a well-defined integral equation which determines the *linear* MMSE estimate of the rate.

1. Introduction

WE CONSIDER a doubly stochastic, time–space Poisson process n^0 with intensity function $\lambda(t, \boldsymbol{r}) = f(t, \boldsymbol{r} - H(t)\boldsymbol{x}_t)$, where $t > 0$ and $\boldsymbol{r} \in \mathbb{R}^2$. Here, f is a known deterministic function; $\boldsymbol{x}_t \in \mathbb{R}^n$ is the solution of an Ito stochastic differential equation, and $H(t)$ is a known deterministic $\mathbb{R}^{2\times n}$-valued function. The process n^0 under consideration counts events which occur in all of $\mathbb{R}^2$; however, suppose that only those events which occur within a region $D \subseteq \mathbb{R}^2$ can be observed. We wish to compute minimum-mean-square-error (MMSE) estimates of $\lambda(t, \boldsymbol{r})$, given our limited observations. In general, this is a difficult problem, and little is known. (See the Remarks in [7] and the references listed there.) When

$$D = \mathbb{R}^2 \quad \text{and} \quad f(t, \boldsymbol{r}) = \exp\left[-\tfrac{1}{2}\boldsymbol{r}^{\mathrm{T}} R(t)^{-1} \boldsymbol{r}\right],$$

for some deterministic positive-definite matrix $R(t)$, we extend a result of Rhodes & Snyder [4] to compute the MMSE estimate of $\lambda(t, \boldsymbol{r})$ explicitly. We also consider *linear* estimates of $\lambda(t, \boldsymbol{r})$ for the same choice of f when $D \neq \mathbb{R}^2$. These filtering problems are frequently encountered in optical communication systems [5, 6], particularly in the context of hypothesis-testing; this issue is discussed in Section 5.

2. Probabilistic setting

Let $\mathfrak{B}^2$ denote the Borel subsets of $\mathbb{R}^2$. Next, if I is any interval of $\mathbb{R}$, let $\mathfrak{B}(I)$ denote the Borel subsets of I. We define $\mathfrak{B}(I) \otimes \mathfrak{B}^2$ to be the smallest σ-algebra containing all sets of the form $E \times A$, such that $E \in \mathfrak{B}(I)$ and $A \in \mathfrak{B}^2$. Let

$(\Omega, \mathcal{F}, \mathrm{p})$ be a probability space on which we let

$$n^0 = (n(B) : B \in \mathfrak{B}(0, \infty) \otimes \mathfrak{B}^2),$$

be a time–space point process. Sometimes, n^0 is called a random point field or a random measure. Here, this means that with each $B \in \mathfrak{B}(0, \infty) \otimes \mathfrak{B}^2$, we associate a nonnegative, integer-valued random variable, $n(B) = n(\omega, B)$; in addition, for each $\omega \in \Omega$, the function $n(\omega, \bullet)$ is assumed to be an integer-valued measure on $\mathfrak{B}(0, \infty) \otimes \mathfrak{B}^2$. We let $\mathcal{F}_t$ represent the times and locations at which points have occurred up to and including time t. More precisely, let $\mathcal{F}_0$ denote the trivial σ-algebra, and for $t > 0$, set

$$\mathcal{F}_t = \sigma\{n(B) : B \in \mathfrak{B}(0, t] \otimes \mathfrak{B}^2\}.$$

Now, let D be a Borel subset of $\mathbb{R}^2$. We take $\mathcal{G}_0$ to be the trivial σ-algebra and, for $t > 0$, we set

$$\mathcal{G}_t = \sigma\{n(B \cap [(0, \infty) \times D]) : B \in \mathfrak{B}(0, t] \otimes \mathfrak{B}^2\}.$$

Note that $\mathcal{G}_t$ represents the history of the point process restricted to the region D, up to time t. We shall refer to $\mathcal{G}_t$ as our 'observations up to time t'. On the same probability space $(\Omega, \mathcal{F}, \mathrm{p})$, let $\boldsymbol{\xi}$ be an n-dimensional Gaussian random vector with known mean $\boldsymbol{m}$ and known positive-definite covariance S. Let $(\boldsymbol{v}_t : t \geq 0)$ be a standard Wiener process independent of $\boldsymbol{\xi}$. We let the n-dimensional process $(\boldsymbol{x}_t : t \geq 0)$ be the solution to the Ito stochastic differential equation

$$\mathrm{d}\boldsymbol{x}_t = F(t)\boldsymbol{x}_t \, \mathrm{d}t + V(t) \, \mathrm{d}\boldsymbol{v}_t, \qquad \boldsymbol{x}_0 = \boldsymbol{\xi}. \tag{2.1}$$

Here F and V are known matrices with appropriate dimensions. We also assume that F and V are piecewise continuous so that a unique solution of (2.1) exists (see Davis [2], pp. 108–111). Let

$$\mathcal{X}_0 \triangleq \sigma\{\boldsymbol{x}_s : 0 \leq s < \infty\}.$$

For $t > 0$, let $\mathcal{X}_t$ denote the smallest σ-algebra containing $\mathcal{F}_t \cup \mathcal{X}_0$. We write this symbolically as

$$\mathcal{X}_t \triangleq \mathcal{F}_t \vee \mathcal{X}_0 \quad (t > 0).$$

We shall assume that n^0 is an $(\mathcal{X}_t : t \geq 0)$-doubly-stochastic time–space Poisson process, with $\mathcal{X}_0$-measurable intensity (see Bremaud [1], pp. 21–23 and 233–238)

$$\lambda(t, \boldsymbol{r}) = f(t, \boldsymbol{r} - H(t)\boldsymbol{x}_t),$$

where $t \in (0, \infty)$, $\boldsymbol{r} \in \mathbb{R}^2$, and $\boldsymbol{x}_t$ is defined by (2.1). Assume that $H : (0, \infty) \to \mathbb{R}^{2 \times n}$ and $f : (0, \infty) \times \mathbb{R}^2 \to (0, \infty)$ are deterministic and known. We further assume that the function

$$\mu(t) \triangleq \int_{\mathbb{R}^2} f(t, \boldsymbol{r}) \, \mathrm{d}\boldsymbol{r}$$

is finite for all $t < \infty$. This means that, for each $t \geq 0$, the process

$$n^t \triangleq (n(B) : B \in \mathfrak{B}(t, \infty) \otimes \mathfrak{B}^2)$$

is a Poisson random field under the measure $\mathrm{p}(\bullet \mid \mathcal{X}_t)$, with rate $\lambda(s, \boldsymbol{r})$, where

$s \in (t, \infty)$ and $\boldsymbol{r} \in \mathbb{R}^2$. This implies the following. First, for $B \in \mathfrak{B}(0, \infty) \otimes \mathfrak{B}^2$, let

$$\lambda(B) \triangleq \int_B \lambda(s, \boldsymbol{r}) \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}s;$$

then if $B \in \mathfrak{B}(t, \infty) \otimes \mathfrak{B}^2$ and n is an arbitrary nonnegative integer,

$$\mathsf{p}[n(B) = n \mid \mathscr{X}_t] = \frac{\lambda(B)^n}{n!} \mathrm{e}^{-\lambda(B)},$$

and hence, for $\theta \in \mathbb{R}$,

$$\mathsf{E}(\mathrm{e}^{\mathrm{j}\theta n(B)} \mid \mathscr{X}_t) = \exp[(\mathrm{e}^{\mathrm{j}\theta} - 1)\lambda(B)].$$

The second implication is that if B_1 and B_2 are disjoint sets in $\mathfrak{B}(t, \infty) \otimes \mathfrak{B}^2$, then the random variables $n(B_1)$ and $n(B_2)$ are independent under the measure $\mathsf{p}(\bullet \mid \mathscr{X}_t)$.

Notation. We let $n_0 \equiv 0$ and, for $t > 0$, $n_t \triangleq n\big((0, t] \times D\big)$.

3. Nonlinear-filtering results

THEOREM 1 *If $D = \mathbb{R}^2$, and if*

$$f(t, \boldsymbol{r}) = \exp[-\tfrac{1}{2}\boldsymbol{r}^{\mathrm{T}} R(t)^{-1} \boldsymbol{r}], \tag{3.1}$$

for some deterministic positive-definite matrix $R(t)$, then

$$\begin{aligned}\hat{\lambda}(t, \boldsymbol{r}) &\triangleq \mathsf{E}[\lambda(t, \boldsymbol{r}) \mid \mathscr{G}_t] = \mathsf{E}[f(t, \boldsymbol{r} - H(t)\boldsymbol{x}_t) \mid \mathscr{G}_t] \\ &= \frac{\sqrt{\det R(t)}}{\sqrt{\det Q_t}} \exp\{-\tfrac{1}{2}[\boldsymbol{r} - H(t)\hat{\boldsymbol{x}}_t]^{\mathrm{T}} Q_t^{-1} [\boldsymbol{r} - H(t)\hat{\boldsymbol{x}}_t]\},\end{aligned}$$

where

$$\hat{\boldsymbol{x}}_t \triangleq \mathsf{E}(\boldsymbol{x}_t \mid \mathscr{G}_t), \qquad \hat{\Sigma}_t \triangleq \mathsf{E}[(\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t)(\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t)^{\mathrm{T}} \mid \mathscr{G}_t] > 0 \quad (\text{p-a.s.}),$$
$$Q_t \triangleq H(t)\hat{\Sigma}_t H(t)^{\mathrm{T}} + R(t),$$

and

$$\mathrm{d}\hat{\boldsymbol{x}}_t = F(t)\hat{\boldsymbol{x}}_t \, \mathrm{d}t + \int_{\mathbb{R}^2} \hat{\Sigma}_{t-} H(t-)^{\mathrm{T}} Q_{t-}^{-1} [\boldsymbol{r} - H(t-)\hat{\boldsymbol{x}}_{t-}] n(\mathrm{d}t \times \mathrm{d}\boldsymbol{r}), \qquad \hat{\boldsymbol{x}}_0 = \boldsymbol{m}, \tag{3.2}$$

$$\mathrm{d}\hat{\Sigma}_t = F(t)\hat{\Sigma}_t \, \mathrm{d}t + \hat{\Sigma}_t F(t)^{\mathrm{T}} \, \mathrm{d}t + V(t)V(t)^{\mathrm{T}} \, \mathrm{d}t - \hat{\Sigma}_{t-} H(t-)^{\mathrm{T}} Q_{t-}^{-1} H(t-) \hat{\Sigma}_{t-} n(\mathrm{d}t \times \mathbb{R}^2),$$
$$\hat{\Sigma}_0 = S. \tag{3.3}$$

Proof. First, since $D = \mathbb{R}^2$, we have $\mathscr{G}_t = \mathscr{F}_t$. It is proved in [4] that the conditional density of $\boldsymbol{x}_t$ given $\mathscr{F}_t$ is Gaussian, with conditional mean $\hat{\boldsymbol{x}}_t$ and conditional covariance $\hat{\Sigma}_t$ (which is positive-definite almost surely because of the assumption that S is positive-definite) satisfying (3.2) and (3.3) above. Let $p_t(\boldsymbol{x})$ denote the conditional distribution of $\boldsymbol{x}_t$ given $\mathscr{F}_t$. Then

$$\begin{aligned}\hat{\lambda}(t, \boldsymbol{r}) &= \int_{\mathbb{R}^n} f\big(t, \boldsymbol{r} - H(t)\boldsymbol{x}\big) \, \mathrm{d}p_t(\boldsymbol{x}) \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{\det \hat{\Sigma}_t}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}[\boldsymbol{r} - H(t)\boldsymbol{x}]^{\mathrm{T}} R(t)^{-1} [\boldsymbol{r} - H(t)\boldsymbol{x}]} \mathrm{e}^{-\frac{1}{2}(\boldsymbol{x} - \hat{\boldsymbol{x}}_t)^{\mathrm{T}} \hat{\Sigma}_t^{-1} (\boldsymbol{x} - \hat{\boldsymbol{x}}_t)} \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.4}$$

At this point one could combine the exponentials above and, after tedious matrix-algebraic calculations, derive the asserted formula for $\hat{\lambda}(t, \boldsymbol{r})$. However, this is not necessary. Let

$$\hat{l}(t, \boldsymbol{\theta}) \triangleq \int_{\mathbb{R}^2} \hat{\lambda}(t, \boldsymbol{r}) \mathrm{e}^{\mathrm{j}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{r}} \,\mathrm{d}\boldsymbol{r} \quad (\boldsymbol{\theta} \in \mathbb{R}^2). \tag{3.5}$$

Substituting equation (3.4) into equation (3.5) and applying Fubini's theorem, we see that

$$\hat{l}(t, \boldsymbol{\theta}) = \tilde{f}(t, \boldsymbol{\theta}) \psi_t(H(t)^{\mathrm{T}} \boldsymbol{\theta}),$$

where

$$\psi_t(\boldsymbol{\eta}) \triangleq \mathsf{E}(\mathrm{e}^{\mathrm{j}\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{x}_t} \mid \mathscr{G}_t) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{j}\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{x}} \,\mathrm{d}p_t(\boldsymbol{x}) \quad (\boldsymbol{\eta} \in \mathbb{R}^n)$$

and

$$\tilde{f}(t, \boldsymbol{\theta}) \triangleq \int_{\mathbb{R}^2} f(t, \boldsymbol{r}) \mathrm{e}^{\mathrm{j}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{r}} \,\mathrm{d}\boldsymbol{r}. \tag{3.6}$$

Since $\boldsymbol{x}_t$ is conditionally Gaussian,

$$\psi_t(\boldsymbol{\eta}) = \exp\left(\mathrm{j}\boldsymbol{\eta}^{\mathrm{T}} \hat{\boldsymbol{x}}_t - \tfrac{1}{2} \boldsymbol{\eta}^{\mathrm{T}} \hat{\Sigma}_t \boldsymbol{\eta}\right).$$

From equations (3.1) and (3.6),

$$\tilde{f}(t, \boldsymbol{\theta}) = 2\pi \sqrt{\det R(t)} \exp\left[-\tfrac{1}{2} \boldsymbol{\theta}^{\mathrm{T}} R(t) \boldsymbol{\theta}\right].$$

Hence,

$$\hat{l}(t, \boldsymbol{\theta}) = 2\pi \sqrt{\det R(t)} \exp\left[\mathrm{j}\boldsymbol{\theta}^{\mathrm{T}} H(t) \hat{\boldsymbol{x}}_t - \tfrac{1}{2} \boldsymbol{\theta}^{\mathrm{T}} Q_t \boldsymbol{\theta}\right].$$

Taking inverse Fourier transforms, we see by inspection that

$$\hat{\lambda}(t, \boldsymbol{r}) = \frac{\sqrt{\det R(t)}}{\sqrt{\det Q_t}} \exp\left\{-\tfrac{1}{2}(\boldsymbol{r} - H(t)\hat{\boldsymbol{x}}_t)^{\mathrm{T}} Q_t^{-1} [\boldsymbol{r} - H(t)\hat{\boldsymbol{x}}_t]\right\}. \quad \square$$

When $D \neq \mathbb{R}^2$ or equation (3.1) does not hold, $p_t(\boldsymbol{x})$ is, in general, not known. This has led us to consider *linear* estimates of $\lambda(t, \boldsymbol{r})$. We discuss this in the next section.

4. Linear-filtering results

We call $\hat{\lambda}_{\mathrm{L}}(t, \boldsymbol{r})$ a *linear* estimate of $\lambda(t, \boldsymbol{r})$ given $\mathscr{G}_t$, if $\hat{\lambda}_{\mathrm{L}}$ can be written in the form

$$\hat{\lambda}_{\mathrm{L}}(t, \boldsymbol{r}) = \int_0^t \int_D h(t, \boldsymbol{r}; \tau, \boldsymbol{\rho})[n(\mathrm{d}\tau \times \mathrm{d}\boldsymbol{\rho}) - \bar{\lambda}(\tau, \boldsymbol{\rho}) \,\mathrm{d}\boldsymbol{\rho} \,\mathrm{d}\tau] + h_0(t, \boldsymbol{r}), \tag{4.1}$$

where h and h_0 are deterministic, and $\bar{\lambda}(t, \boldsymbol{r}) \triangleq \mathsf{E}\lambda(t, \boldsymbol{r})$. We wish to choose h and h_0 to minimize

$$\mathsf{E}\,|\lambda(t, \boldsymbol{r}) - \hat{\lambda}_{\mathrm{L}}(t, \boldsymbol{r})|^2. \tag{4.2}$$

This leads us to the following theorem.

THEOREM 2 *Let $\hat{\lambda}_L(t, \boldsymbol{r})$ be given by* (4.1). *Under the conditions outlined in Section* 2, *the quantity in* (4.2) *will be minimized if $h_0(t, \boldsymbol{r}) = \bar{\lambda}(t, \boldsymbol{r})$, and if h satisfies*

$$\gamma(t, \boldsymbol{r}; \tau, \boldsymbol{\rho}) = \int_0^t \int_D h(t, \boldsymbol{r}; \sigma, \zeta)\gamma(\sigma, \zeta; \tau, \boldsymbol{\rho})\, d\zeta\, d\sigma + h(t, \boldsymbol{r}; \tau, \boldsymbol{\rho})\bar{\lambda}(\tau, \boldsymbol{\rho}), \quad (4.3)$$

where

$$\gamma(t, \boldsymbol{r}; \tau, \boldsymbol{\rho}) \triangleq \operatorname{cov}[\lambda(t, \boldsymbol{r}), \lambda(\tau, \boldsymbol{\rho})].$$

If $f(t, \boldsymbol{r})$ is given by (3.1), *then*

$$\bar{\lambda}(t, \boldsymbol{r}) = \frac{\sqrt{\det R(t)}}{\sqrt{\det Q(t)}} \exp\{-\tfrac{1}{2}[\boldsymbol{r} - H(t)\bar{\boldsymbol{x}}(t)]^{\mathrm{T}} Q(t)^{-1}[\boldsymbol{r} - H(t)\bar{\boldsymbol{x}}(t)]\}, \quad (4.4)$$

where

$$\bar{\boldsymbol{x}}(t) \triangleq \mathsf{E}\boldsymbol{x}_t, \qquad \Sigma(t) \triangleq \operatorname{cov} \boldsymbol{x}_t, \qquad Q(t) \triangleq H(t)\Sigma(t)H(t)^{\mathrm{T}} + R(t).$$

Furthermore,

$$\gamma(t, \boldsymbol{r}; \tau, \boldsymbol{\rho}) + \bar{\lambda}(t, \boldsymbol{r})\bar{\lambda}(\tau, \boldsymbol{\rho}) = \sqrt{\frac{\det R(t) \det R(\tau)}{\det Q(t, \tau)}}$$
$$\times \exp\left\{-\tfrac{1}{2}\left(\begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{\rho} \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \bar{\boldsymbol{x}}(\tau) \end{bmatrix}\right)^{\mathrm{T}}\right.$$
$$\left. \times Q(t, \tau)^{-1}\left(\begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{\rho} \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \bar{\boldsymbol{x}}(\tau) \end{bmatrix}\right)\right\}, \quad (4.5)$$

where

$$\Sigma(t, \tau) \triangleq \operatorname{cov}(\boldsymbol{x}_t, \boldsymbol{x}_\tau), \qquad Q(t, \tau) \triangleq \begin{bmatrix} Q(t) & H(t)\Sigma(t, \tau)H(\tau)^{\mathrm{T}} \\ H(\tau)\Sigma(\tau, t)H(t)^{\mathrm{T}} & Q(\tau) \end{bmatrix}.$$

Proof. The fact that we should set $h_0(t, \boldsymbol{r}) = \bar{\lambda}(t, \boldsymbol{r})$ and that h should satisfy (4.3) is proved in Grandell [3]. Proceeding further, we make the following observations. Recall that

$$d\boldsymbol{x}_t = F(t)\boldsymbol{x}_t\, dt + V(t)\, d\boldsymbol{v}_t, \qquad \boldsymbol{x}_0 = \boldsymbol{\xi}. \quad (4.6)$$

Let $\Phi(t_2, t_1)$ be the transition matrix corresponding to $F(t)$. Then

$$\bar{\boldsymbol{x}}(t) = \Phi(t, 0)\boldsymbol{m}, \quad (4.7)$$

and

$$\Sigma(t, \tau) = \Phi(t, 0)S\Phi(\tau, 0)^{\mathrm{T}} + \int_0^{\min(t, \tau)} \Phi(t, s)V(s)V(s)^{\mathrm{T}}\Phi(\tau, s)^{\mathrm{T}}\, ds.$$

Note that $\Sigma(t) = \Sigma(t, t)$.

To compute $\bar{\lambda}(t, \boldsymbol{r}) = \mathsf{E}\lambda(t, \boldsymbol{r})$, observe that $\boldsymbol{x}_t$ is Gaussian with mean $\bar{\boldsymbol{x}}(t)$ and covariance $\Sigma(t)$. By using the same procedure as in the proof of Theorem 1, equation (4.4) is immediate.

The computation of (4.5) is similar, but requires some judicious preliminary algebra. First, observe that $\gamma(t, \boldsymbol{r}; \tau, \boldsymbol{\rho}) + \bar{\lambda}(t, \boldsymbol{r})\bar{\lambda}(\tau, \boldsymbol{\rho})$ is just another way of writing $\mathsf{E}[\lambda(t, \boldsymbol{r})\lambda(\tau, \boldsymbol{\rho})]$. Next, rewrite $\lambda(t, \boldsymbol{r})\lambda(\tau, \boldsymbol{\rho})$ as

$$\exp\left\{-\tfrac{1}{2}\left(\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{\rho}\end{bmatrix} - \begin{bmatrix}H(t) & 0\\ 0 & H(\tau)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_t\\ \boldsymbol{x}_\tau\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}R(t)^{-1} & 0\\ 0 & R(\tau)^{-1}\end{bmatrix}\left(\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{\rho}\end{bmatrix} - \begin{bmatrix}H(t) & 0\\ 0 & H(\tau)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_t\\ \boldsymbol{x}_\tau\end{bmatrix}\right)\right\},$$

which is equal to

$$\exp\left\{-\tfrac{1}{2}\left(\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{\rho}\end{bmatrix} - \begin{bmatrix}H(t) & 0\\ 0 & H(\tau)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_t\\ \boldsymbol{x}_\tau\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}R(t) & 0\\ 0 & R(\tau)\end{bmatrix}^{-1}\left(\begin{bmatrix}\boldsymbol{r}\\ \boldsymbol{\rho}\end{bmatrix} - \begin{bmatrix}H(t) & 0\\ 0 & H(\tau)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_t\\ \boldsymbol{x}_\tau\end{bmatrix}\right)\right\}.$$

Because $(\boldsymbol{x}_t : t \geqslant 0)$ is a Gaussian process, $\begin{bmatrix}\boldsymbol{x}_t\\ \boldsymbol{x}_\tau\end{bmatrix}$ is a Gaussian random vector with mean $\begin{bmatrix}\bar{\boldsymbol{x}}(t)\\ \bar{\boldsymbol{x}}(\tau)\end{bmatrix}$ and covariance $\begin{bmatrix}\Sigma(t) & \Sigma(t, \tau)\\ \Sigma(\tau, t) & \Sigma(\tau)\end{bmatrix}$. By the same reasoning used to deduce (4.4), equation (4.5) also follows. □

5. Discussion

The filtering problems considered above often arise in the design and implementation of receivers for optical communication systems. Typically, a binary message source is used by a transmitter to select the modulation of the intensity of a laser beam in accordance with whether a '0' or a '1' is to be sent. The laser beam travels to a receiver and strikes its photodetector. We assume that the laser beam has an intensity profile of the form

$$v_i(t)f(t, \boldsymbol{r}) \quad (i = 0, 1).$$

Here, $v_i(t)$ is a known, deterministic function, where $i = 0$ or 1 has been selected by the transmitter.

We model the surface of the receiver's photodetector as $\mathbb{R}^2$. If the receiver, for example, is subject to vibrations, the centre of the spot of laser light may wander randomly over the photodetector surface ([6]). We assume, as in [6], that the centre of the spot of laser light is given by $H(t)\boldsymbol{x}_t \in \mathbb{R}^2$. The output of photoelectrons from the photodetector is modelled by the process n^0, with stochastic intensity now given by

$$\lambda_i(t, \boldsymbol{r}) = v_i(t)f\big(t, \boldsymbol{r} - H(t)\boldsymbol{x}_t\big). \tag{5.1}$$

Of course, an actual photodetector does not have an infinite photosensitive surface. We account for this fact by assuming that only those photoelectrons which occur in a region $D \subseteq \mathbb{R}^2$ are observed. For example, in this setting, D might be a square or a circle centred at the origin. After observing photoelectrons occurring in D during some time interval $[0, T]$, a decision as to whether a 0 or a 1 was sent has to be made, based on one of the estimates $\hat{\lambda}_i(t, \boldsymbol{r})$ or $\hat{\lambda}_{i,\mathrm{L}}(t, \boldsymbol{r})$. As an example of a decoding scheme, we could use the likelihood-ratio test

$$L_T \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

to make the decision, using the minimum probability of error as cost criterion and assuming equiprobable hypotheses (Snyder [5], section 2.5). The likelihood ratio L_T is given by ([5], pp. 471–476)

$$L_T = \frac{(\prod_{j=1}^{n_T} \hat{\lambda}_1(t_j, \boldsymbol{r}_j)) \exp(-\int_0^T \int_D \hat{\lambda}_1(s, \boldsymbol{r})\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}s)}{(\prod_{j=1}^{n_T} \hat{\lambda}_0(t_j, \boldsymbol{r}_j)) \exp(-\int_0^T \int_D \hat{\lambda}_0(s, \boldsymbol{r})\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}s)}, \tag{5.2}$$

where t_j and $\boldsymbol{r}_j$ are respectively the time and the location of the jth photoevent in the region D, and we adopt the convention that when $n_T = 0$, the empty product factors are omitted, i.e. taken to be unity. Here, of course,

$$\hat{\lambda}_i(t, \boldsymbol{r}) \triangleq \mathsf{E}_i[\lambda_i(t, \boldsymbol{r}) \mid \mathscr{G}_t] \quad (i = 0, 1).$$

where E_i denotes conditional expectation under hypothesis i. Now, using (5.1), equation (5.2) simplifies to

$$L_T = \prod_{j=1}^{n_T} \frac{v_1(t_j)\hat{f}_1(t_j, \boldsymbol{r}_j)}{v_0(t_j)\hat{f}_0(t_j, \boldsymbol{r}_j)} \exp\left(-\int_0^T \int_D [\hat{\lambda}_1(s, \boldsymbol{r}) - \hat{\lambda}_0(s, \boldsymbol{r})]\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}s\right). \tag{5.3}$$

where

$$\hat{f}_i(t, \boldsymbol{r}) \triangleq \mathsf{E}_i[f(t, \boldsymbol{r} - H(t)\boldsymbol{x}_t \mid \mathscr{G}_t] \quad (i = 0, 1).$$

In the general case, $D \neq \mathbb{R}^2$, the estimate $\hat{\lambda}_i(t, \boldsymbol{r})$ is not known, and hence L_T cannot be computed. However, when $D = \mathbb{R}^2$, observe that

$$\begin{aligned}\int_D [\hat{\lambda}_1(s, \boldsymbol{r}) - \hat{\lambda}_0(s, \boldsymbol{r})]\, \mathrm{d}\boldsymbol{r} &= \mathsf{E}_1\left(\int_{\mathbb{R}^2} \lambda_1(s, \boldsymbol{r})\, \mathrm{d}\boldsymbol{r} \,\middle|\, \mathscr{G}_s\right) - \mathsf{E}_0\left(\int_{\mathbb{R}^2} \lambda_0(s, \boldsymbol{r})\, \mathrm{d}\boldsymbol{r} \,\middle|\, \mathscr{G}_s\right) \\ &= \mu(s)[v_1(s) - v_0(s)].\end{aligned} \tag{5.4}$$

In equation (5.4) we used the fact that for all $\boldsymbol{r}_0 \in \mathbb{R}^2$,

$$\mu(s) \triangleq \int_{\mathbb{R}^2} f(s, \boldsymbol{r})\, \mathrm{d}\boldsymbol{r} = \int_{\mathbb{R}^2} f(s, \boldsymbol{r} - \boldsymbol{r}_0)\, \mathrm{d}\boldsymbol{r}.$$

Thus, when $D = \mathbb{R}^2$, (5.3) becomes

$$L_T = \prod_{j=1}^{n_T} \frac{v_1(t_j)\hat{f}_1(t_j, \boldsymbol{r}_j)}{v_0(t_j)\hat{f}_0(t_j, \boldsymbol{r}_j)} \exp\left(-\int_0^T \mu(s)[v_1(s) - v_0(s)]\, \mathrm{d}s\right). \tag{5.5}$$

where $\hat{f}_i$ can be determined using Theorem 1.

We next consider the following theorem.

THEOREM 3 *The random field*

$$m' \triangleq (n(E \times \mathbb{R}^2) : E \in \mathfrak{B}(t, \infty))$$

is independent of the σ-algebra $\mathscr{X}_t$.

Proof. To prove that m' is independent of $\mathscr{X}_t$, it is sufficient to show that the conditional characteristic function of $n(E \times \mathbb{R}^2)$ is deterministic for $E \in \mathfrak{B}(t, \infty)$. Now, from the assumption that n^0 is an $(\mathscr{X}_t : t \geq 0)$-doubly-stochastic time–space Poisson process, it follows immediately that, for $\theta \in \mathbb{R}$,

$$\mathsf{E}_i(\mathrm{e}^{\mathrm{j}\theta n(E\times\mathbb{R}^2)} \mid \mathscr{X}_t) = \exp\left((\mathrm{e}^{\mathrm{j}\theta} - 1)\int_E\int_{\mathbb{R}^2} \lambda_i(s, \boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}s\right)$$

$$= \exp\left((\mathrm{e}^{\mathrm{j}\theta} - 1)\int_E \nu_i(s)\int_{\mathbb{R}^2} f\big(s, \boldsymbol{r} - H(s)\boldsymbol{x}_s\big)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}s\right)$$

$$= \exp\left((\mathrm{e}^{\mathrm{j}\theta} - 1)\int_E \nu_i(s)\mu(s)\,\mathrm{d}s\right).$$

Hence m^t is independent of $\mathscr{X}_t$. □

It follows from equation (5.5) and Theorem 3 that, for all $t \geq 0$, the random variable L_t is independent of the σ-algebra $\mathscr{X}_0$.

If we replace equation (2.1) by

$$\mathrm{d}\boldsymbol{x}_t = F(t)\boldsymbol{x}_t\,\mathrm{d}t + G(t)\boldsymbol{u}_t\,\mathrm{d}t + V(t)\,\mathrm{d}\boldsymbol{v}_t; \qquad \boldsymbol{x}_0 = \boldsymbol{\xi}, \tag{5.6}$$

where $(\boldsymbol{u}_t : t \geq 0)$ is predictable with respect to $(\mathscr{G}_t : t \geq 0)$ and $G(t)$ is a known matrix with approximate dimensions, then most of the above results hold with only minor modifications. The term $G(t)\boldsymbol{u}_t$ in (5.6) is interpreted as a control signal driven by the output of the photodetector. Since $H(t)\boldsymbol{x}_t$ represents the centre of the spot of laser light striking the receiver, one might try to use $G(t)\boldsymbol{u}_t$ to drive $\boldsymbol{x}_t$ to the origin. This problem is addressed in [4]. If (2.1) is replaced by (5.6), Theorem 1 still holds except that equation (3.2) must be replaced by

$$\mathrm{d}\hat{\boldsymbol{x}}_t = F(t)\hat{\boldsymbol{x}}_t\,\mathrm{d}t + G(t)\boldsymbol{u}_t\,\mathrm{d}t + \int_{\mathbb{R}^2} \hat{\Sigma}_{t-}H(t-)^{\mathrm{T}}Q_{t-}^{-1}[\boldsymbol{r} - H(t-)\hat{\boldsymbol{x}}_{t-}]n(\mathrm{d}t \times \mathrm{d}\boldsymbol{r}),$$
$$\hat{\boldsymbol{x}}_0 = \boldsymbol{m}.$$

If $\boldsymbol{u}_t = \boldsymbol{u}(t)$ for some deterministic control $(\boldsymbol{u}(t) : t \geq 0)$, then Theorem 2 holds; of course, (4.6) becomes (5.6) and (4.7) is replaced by

$$\bar{\boldsymbol{x}}(t) = \Phi(t, 0)\boldsymbol{m} + \int_0^t \Phi(t, s)G(s)\boldsymbol{u}(s)\,\mathrm{d}s.$$

In addition, the results of the preceding paragraphs of Section 5, including Theorem 3, are unchanged by substituting equation (5.6) for equation (2.1).

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